WS 8.4, 8.6

1 Problem 1

1. Where applicable in $(a)-(c)$, carry out the necessary division in preparation for integration. If division is not needed, give a reason why it is not needed.

(a) $\frac{x^2}{x^2-x+1}$
(b) $\frac{x-1}{x^2-x+1}$
(a) $\frac{x^4+4}{x^2+1}$

(a) The degree on top is bigger, so we should perform a division [to divide, successively multiply the divisor (x^2-x+1) by an appropriate Cx^n (this goes on top each time) in order to match the highest term of what's inside the division sign (x^2, then $x-1$) then subtract, cancelling it, and giving you a lower degree inside the division]:

$$\begin{array}{r|r} & 1 \\ \hline x^2-x+1) & x^2 \\ & -x^2+x-1 \\ \hline & x-1 \end{array}$$

Thus $\frac{x^2}{x^2-x+1} = 1 + \frac{x-1}{x^2-x+1}$.

(b) Here polynomial division is unnecessary since the degree on the top is less than the degree on the bottom.

(c) The degree on top is bigger, so we should perform a division:

$$\begin{array}{r|r} & x^2-1 \\ \hline x^2+1) & x^4 \qquad +4 \\ & -x^4-x^2 \qquad \\ \hline & -x^2+4 \\ & x^2+1 \\ \hline & 5 \end{array}$$

Thus $\frac{x^4+4}{x^2+1} = (x^2-1) + \frac{5}{x^2+1}$.

2 Problem 2

2. In the following, write the rational function as a combination of factors like those appearing in (4) and/or (5) of Section 8.4, without finding the numerical values of the constants $A_1, ..., B_1, ..., C_1, ...$

(a) $\frac{6x^2-4x+1}{(x-2)^3}$

(b) $\frac{x^3+2x^2+3x+4}{x^4-16}$

(a) We'll set up our partial fraction decomposition since the denominator is already factored: $\frac{6x^2-4x+1}{(x-2)^3} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3}$. Now multiply both sides by the denominator $(x-2)^3$:

$$6x^2 - 4x + 1 = A(x-2)^2 + B(x-2) + C$$
$$\implies 6x^2 - 4x + 1 = Ax^2 - 2Ax + 4A + Bx - 2B + C = Ax^2 + (B-2A)x + 4A + C - 2B$$

Then equating coefficients, we gain the following system of equations:

$$A = 6$$
$$B - 2A = -4$$
$$4A + C - 2B = 1$$

We use the first two to solve for B: $B - 2(6) = -4 \implies B = 8$.

Now we can use the last equation to solve for C: $4(6) + C - 2(8) = 1$ so $C = 1 + 16 - 24 = -7$.

Thus

$$\frac{6x^2-4x+1}{(x-2)^3} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} = \frac{6}{x-2} + \frac{8}{(x-2)^2} - \frac{7}{(x-2)^3}.$$

Thus we're done.

(b) Note we don't need to do any division, since the degree on top is smaller than 4.

Now, we factor denominator, using difference of perfect squares $a^2 - b^2 = (a+b)(a-b)$: $\frac{x^3+2x^2+3x+4}{x^4-16} = \frac{x^3+2x^2+3x+4}{(x^2-4)(x^2+4)} = \frac{x^3+2x^2+3x+4}{(x+2)(x-2)(x^2+4)}$. Note x^2+4 is an irreducible quadratic since it has no real roots (it's always positive).

We set up our partial fraction decomposition: $\frac{x^3+2x^2+3x+4}{(x+2)(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+4}$.

Now multiply both sides by the denominator $x^4 - 16$:

$$x^3 + 2x^2 + 3x + 4 = A(x+2)(x^2+4) + B(x-2)(x^2+4) + (Cx+D)(x-2)(x+2)$$

Expanding this gets messy so we'll use our shortcuts: plugging in $x = -2$, $x = 2$.

Plugging in $x = -2$, the equation tells us that $(-2)^3 + 2(-2)^2 + 3(-2) + 4 = 0 + B(-4)((-2)^2 + 4) + 0$, or $-24B = -2$, or $B = \frac{1}{12}$.

Plugging in $x = 2$, the equation tells us that $2^3 + 2(2^2) + 3(2) + 4 = A(2+2)(2^2+4) + 0 + 0$ or $24A = 26$, so $A = \frac{13}{12}$.

To find C, there's another shortcut to take: collect the cubed powers on the right. Notice that $(Cx+D)(x-2)(x+2)$ has highest term Cx^3 when we expand it. Note also $A(x+2)(x^2+4)$ has highest term Ax^3 when we expand it, and similarly $B(x-2)(x^2+4)$ has Bx^3. Thus the coefficient on x^3 on the right is $A+B+C$. Equate this to the left hand side's coefficient of x^3, so that $1 = A+B+C$. Thus $\frac{13}{12} + \frac{1}{12} + C = 1$. Thus $C = \frac{-2}{12} = \frac{-1}{6}$.

To find D we can take another shortcut. Look at the constant terms on the right: we'll have $8A$ from $A(x+2)(x^2+4)$, $-8B$ from $B(x-2)(x^2+4)$, and $-4D$ from $(Cx+D)(x-2)(x+2)$. Equate this to the left side: $4 = 8A - 8B - 4D$, so $1 = 2(A-B) - D$. Then $D = 2(A-B) - 1$, so $D = 2(\frac{13}{12} - \frac{1}{12}) - 1 = 2 - 1 = 1$.

3 Problem 3

(a) Show that $\frac{x^2}{(x^2-4)^2} = \frac{1}{4}(\frac{1}{x-2} + \frac{1}{x+2})^2$.

(b) Use partial fractions to solve $\int \frac{x^2}{(x^2-4)^2}\,dx$

(c) Could the integral in (b) be solved by trigonometric substitution? Explain your answer.

(d) Find $\int \frac{\sqrt{x+4}}{x^2}\,dx$. (Hint: Let $u = \sqrt{x+4}$, and then use the result in part (b)!)

(a) Show that $\frac{x^2}{(x^2-4)^2} = \frac{1}{4}(\frac{1}{x-2} + \frac{1}{x+2})^2$.

We'll use partial fractions: we factor $(x^2-4)^2 = (x+2)^2(x-2)^2$ and set up our decomposition of $\frac{x^2}{(x^2-4)^2}$:

$$\frac{x^2}{(x+2)^2(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2}.$$

Multiply both sides by the denominator on the left $(x-2)^2(x+2)^2$ to obtain

$$x^2 = A(x-2)(x+2)^2 + B(x+2)^2 + C(x+2)(x-2)^2 + D(x-2)^2$$

Again this is nasty to expand so we'll take the shortcuts of setting $x = -2$, $x = 2$. Then for $x = -2$, we have $(-2)^2 = D(-4)^2$ so $D = \frac{1}{4}$. For $x = 2$, we'll have $2^2 = B(4)^2$ so $B = \frac{1}{4}$ as well. To find A and C, we'll use our shortcuts.

We find the highest terms on the right: $A(x-2)(x+2)^2$ and $C(x+2)(x-2)^2$ are the only things on the right with x^3. Then we have $Ax^3 + Cx^3 = (A+C)x^3 = 0$ since the left side has no x^3 terms. Thus $A = -C$.

Now we find the lowest terms on the right: we'll have $-8A$ from $A(x-2)(x+2)^2$, $4B$ from $B(x+2)^2$, $8C$ from $C(x+2)(x-2)^2$, and $4D$ from $D(x-2)^2$. These will combine to 0 since there's no constants on the left: $-8A+4B+8C+4D = 0$. We'll plug in our values for $B = D = \frac{1}{4}$ so $-8A+1+8C+1 = 0$, or $8(C-A) = -2$.

Since $A = -C$, $C - A = 2C$. Then $8(2C) = -2$ so $C = \frac{-1}{8}$, and $A = \frac{1}{8}$ since $A = -C$.

Thus,

$$\frac{x^2}{(x+2)^2(x-2)^2} = \frac{1/8}{x-2} + \frac{1/4}{(x-2)^2} + \frac{-1/8}{x+2} + \frac{1/4}{(x+2)^2}.$$

Note that $\frac{1/8}{x-2} + \frac{-1/8}{x+2} = \frac{(1/8)(x+2)+(-1/8)(x-2)}{(x-2)(x+2)} = \frac{1/4+1/4}{x^2-4} = \frac{1/2}{x^2-4}$. Then $\frac{x^2}{(x+2)^2(x-2)^2} = \frac{1/4}{(x-2)^2} + \frac{1/2}{x^2-4} + \frac{1/4}{(x+2)^2}$. This is the same as

$$\frac{1}{4}(\frac{1}{x-2} + \frac{1}{x+2})^2 = \frac{1}{4}(\frac{1}{(x-2)^2} + \frac{2}{(x-2)(x+2)} + \frac{1}{(x+2)^2}) = \frac{1/4}{(x-2)^2} + \frac{1/2}{x^2-4} + \frac{1/4}{(x+2)^2}$$

Thus $\frac{x^2}{(x^2-4)^2} = \frac{1}{4}(\frac{1}{x-2} + \frac{1}{x+2})^2$ and we're done.

(b) Use partial fractions to solve $\int \frac{x^2}{(x^2-4)^2}\,dx$

We'll use the decomposition we just found:

$$\int \frac{x^2}{(x+2)^2(x-2)^2}\,dx = \int \frac{1/8}{x-2} + \frac{1/4}{(x-2)^2} + \frac{-1/8}{x+2} + \frac{1/4}{(x+2)^2}\,dx \tag{1}$$

$$= (1/8)\ln(x-2) + (1/4)\frac{-1}{x-2} - (1/8)\ln(x+2) + (1/4)\frac{-1}{x+2} + C \tag{2}$$

$$= (1/8)\ln(\frac{x-2}{x+2}) - (\frac{1}{4x-8} + \frac{1}{4x+8}) + C = (1/8)\ln(\frac{x-2}{x+2}) - \frac{x}{2x^2-8} + C \tag{3}$$

Thus we're done.
(c) Let's try doing a trig sub for $x = a\sec u$, so $x = 2\sec u$, $dx = 2\sec u\tan u\,du$. Then,

$$\int \frac{x^2}{(x^2-4)^2}\,dx = \int \frac{4\sec^2 u \cdot 2\sec u\tan u}{(4\sec^2 u - 4)^2}\,du \tag{4}$$

$$= \int \frac{8\sec^3 u\tan u}{(4\tan^2 u)^2}\,du \tag{5}$$

$$= \int \frac{8\sec^3 u\tan u}{16\tan^4 u}\,du \tag{6}$$

$$= \int \frac{\sec^3 u}{2\tan^3 u}\,du \tag{7}$$

$$= \int \frac{1/\cos^3 u}{2\sin^3 u/\cos^3 u}\,du \tag{8}$$

$$= \int \frac{1}{2\sin^3 u}\,du \tag{9}$$

$$= \int (1/2)\csc^3 u\,du \tag{10}$$

$$= (1/2)(-(1/2)\csc u\cot u - (1/2)\ln(\csc u + \cot u)) + C \tag{11}$$

$$= -(1/4)\frac{x}{\sqrt{x^2-4}} \cdot \frac{2}{\sqrt{x^2-4}} - (1/4)\ln(\frac{x}{\sqrt{x^2-4}} + \frac{2}{\sqrt{x^2-4}}) + C \tag{12}$$

$$= -\frac{x}{2x^2-8} - (1/4)\ln(\frac{\sqrt{(x+2)^2}}{\sqrt{x^2-4}}) + C \tag{13}$$

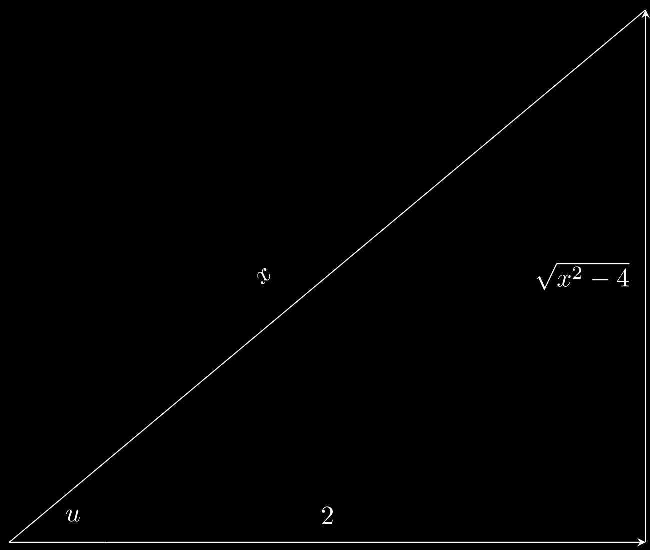
$$= -\frac{x}{2x^2-8} - (1/4)\ln(\sqrt{\frac{(x+2)^2}{(x-2)(x+2)}}) + C \tag{14}$$

$$= -\frac{x}{2x^2-8} - (1/8)\ln(\frac{x+2}{x-2}) + C \tag{15}$$

$$\tag{16}$$

since we've seen $\int \csc^3 x\,dx = -(1/2)\csc x\cot x - (1/2)\ln|\csc x + \cot x| + C$. It can be solved by trig subs, but it's more complicated.

To do the backsubs to go back to x's from u's, we used $\frac{x}{2} = \sec u = \frac{\text{hyp}}{\text{adj}}$ defining a triangle:



Thus $\csc u = \frac{x}{\sqrt{x^2-4}} = \frac{\text{hyp}}{\text{opp}}$, $\cot u = \frac{2}{\sqrt{x^2-4}} = \frac{\text{adj}}{\text{opp}}$.

(d) Find $\int \frac{\sqrt{x+4}}{x^2}\, dx$. (Hint: Let $u = x + 4$, and then use the result in part (b)!)

When we set $u = \sqrt{x+4}$, then $du = -(1/2)\frac{1}{\sqrt{x+4}}\, dx$ so $-2u\, du = dx$ and $x = u^2 - 4$. Then by part (b), we have:

$\int \frac{\sqrt{x+4}}{x^2}\, dx = \int \frac{u \cdot -2u\, du}{(u^2-4)^2} = (-2)(-\frac{u}{2u^2-8} - (1/8)\ln(\frac{u+2}{u-2}) + C) = \frac{2\sqrt{x+4}}{2(\sqrt{x+4})^2-8} + (1/4)\ln(\frac{\sqrt{x+4}+2}{\sqrt{x+4}-2}) + C.$

We'll write this as

$$\frac{\sqrt{x+4}}{x} + (1/4)(\ln(\sqrt{x+4}+2) - \ln(\sqrt{x+4}-2)) + C$$

4 Problem 4

(a) In a complete sentence, tell why there needs to be an *even* number of subintervals in Simpon's Rule.

(b) Does there need to be an even number of subintervals in the Trapezoidal Rule? Explain your answer in a complete sentence.

(a) We need an even number of subintervals of $[a, b]$ in Simpson's rule because parabolas are determined by 3 points: so $a = x_0, x_1, x_2$ determine the first parabola, x_2, x_3, x_4 determine the second, so $x_{2k-2}, x_{2k-1}, x_{2k}$ determine the k^{th} one, and so on until we reach our last point $x_{2k} = x_n = b$. Then the number of subintervals $[x_0, x_1], [x_2, x_3], ..., [x_{2k-1}, x_{2k}] = [x_{n-1}, x_n]$ is even since $n = 2k$ must be even.

(b) There does not need to be an even number of subintervals in the Trapezoidal rule, we just need at least 2 points. If we have 3 points (an odd number) x_0, x_1, x_2, we have two trapezoids, one determined by $(x_0, 0), (x_0, f(x_0)), (x_1, 0), (x_1, f(x_1))$ and one determined by $(x_1, 0), (x_1, f(x_1)), (x_2, 0), (x_2, f(x_2))$.

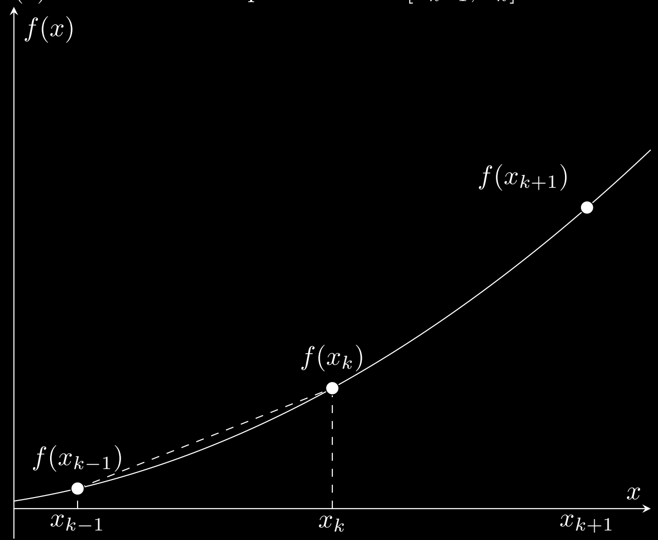
5 Problem 5

Suppose f is continuous and positive on the interval $[x_{k-1}, x_{k+1}]$ and let $x_{k-1} < x_k < x_{k+1}$.

(a) Draw the trapezoid above the interval $[x_{k-1}, x_k]$, with vertices $(x_{k-1}, 0), (x_k, 0), (x_k, f(x_k))$, and $(x_{k-1}, f(x_{k-1}))$. Then find the area A of the trapezoid.

(b) Using the result of (a), find the sum A_1 of the areas of two trapezoids, the one trapezoid above the interval $[x_{k-1}, x_k]$ and the other trapezoid above the interval $[x_k, x_{k+1}]$. Draw a picture for the two trapezoids.

(a) We draw the trapezoid above $[x_{k-1}, x_k]$ with those vertices:



Note that the area of this trapezoid is the area of a rectangle with base x_{k-1} to x_k and height $f(x_{k-1})$ plus the area of a triangle of height $f(x_k) - f(x_{k-1})$ and base x_{k-1} to x_k. Then

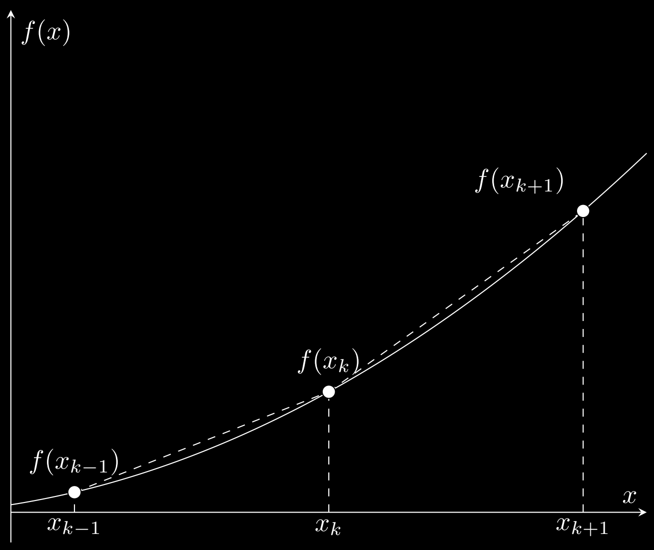
$$A = (x_k - x_{k-1}) \cdot f(x_{k-1}) + \frac{1}{2} \cdot (f(x_k) - f(x_{k-1})) \cdot (x_k - x_{k-1}) \tag{17}$$
$$= f(x_{k-1}) \cdot (x_k - x_{k-1}) + \frac{1}{2} f(x_k) \cdot (x_k - x_{k-1}) - \frac{1}{2} f(x_{k-1}) \cdot (x_k - x_{k-1}) \tag{18}$$
$$= \frac{1}{2} f(x_{k-1}) \cdot (x_k - x_{k-1}) + \frac{1}{2} f(x_k) \cdot (x_k - x_{k-1}) \tag{19}$$
$$= \frac{1}{2} (f(x_k) + f(x_{k-1}))(x_k - x_{k-1}) \tag{20}$$

This is the average height $\frac{f(x_k)+f(x_{k-1})}{2}$ times the base.

(b) Using (a), we have that

$$A_1 = \frac{1}{2}(f(x_k) + f(x_{k-1}))(x_k - x_{k-1}) + \frac{1}{2}(f(x_{k+1}) + f(x_k))(x_{k+1} - x_k) \tag{21}$$
$$= \frac{1}{2} f(x_{k-1})(x_k - x_{k-1}) + \frac{1}{2} f(x_k)(x_k - x_{k-1}) + \frac{1}{2} f(x_k)(x_{k+1} - x_k) + \frac{1}{2} f(x_{k+1})(x_{k+1} - x_k) \tag{22}$$
$$\tag{23}$$

Notice that if $h = x_{k+1} - x_k = x_k - x_{k-1}$, this equation factors as $h/2 \cdot (f(x_{k-1}) + 2f(x_k) + f(x_{k+1}))$. To draw a picture for the two trapezoids:


$f(x)$
$f(x_{k+1})$
$f(x_k)$
$f(x_{k-1})$
x
x_{k-1}
x_k
x_{k+1}